Temperature of a Glass Plate Exposed to the Night Sky
or
Why Does Dew Form so Readily on My Telescope's Corrector Plate

Input Data (MKS system)

T_{airF}	Air temperature Fahrenheit
T_{gndF}	ground temperature Fahrenheit
T_{sky}	sky temperature Kelvin
σ	Stefan-Boltzmann constant
A	Area of glass plate (taken as 1)
k	thermal conductivity of glass
h	convection heat transfer coefficient of air
width	width (thickness) of glass plate suspended above the ground

$T_{airF} := 40$ $T_{gndF} := 45$ $T_{sky} := 203.15$ Kelvin (T_{sky} is about -70 Celsius)

$h := 10$ $A := 1$ $\sigma := 5.669 \cdot 10^{-8}$ $k := .78$

$width := .000000001$ Meters (Really thin glass plate!)

Temperature conversion formulas:

$$FtoK(T_F) := \left(\frac{5}{9}\right) \cdot (T_F - 32) + 273.15$$ Fahrenheit to Kelvin

$$KtoF(T_F) := \left(\frac{9}{5}\right) \cdot (T_F - 273.15) + 32$$ Kelvin to Fahrenheit

So

$T_{air} := FtoK(T_{airF})$ $T_{air} = 277.594$ Kelvin

$T_{gnd} := FtoK(T_{gndF})$ $T_{gnd} = 280.372$ Kelvin

VERY THIN PLATE

The following is a formula and calculation of the Kelvin temperature for the top of a very thin plate which can be used as a rough check on results obtained later for the more general formula developed below for a plate with finite thickness.

We will see later that a plate with finite thickness retards the heat flow and causes the top surface of a thick plate to become colder than the top surface of an extremely thin plate.

The formula immediately below is from Reference 3.

$$T_{cp} := \mathrm{root}\left[\begin{array}{l}\sigma\cdot A\cdot\left(T_{cp}{}^{4} - T_{sky}{}^{4}\right)\ \ldots \\ +\,\sigma\cdot A\cdot\left(T_{cp}{}^{4} - T_{gnd}{}^{4}\right)\ \ldots \\ +\,-2\cdot h\cdot A\cdot\left(T_{air} - T_{cp}\right)\end{array}, T_{cp}, T_{air} - 40, T_{air} + 40\right] \ \mathrm{float}, 5 \ \rightarrow 269.87$$

Or $\mathrm{KtoF}\left(T_{cp}\right) = 26.096$ Fahrenheit

DERIVATION OF HEAT EQUATIONS FOR A GLASS PLATE OF FINITE THICKNESS

Now here are the equations for a thick glass plate using heat transfer equations of conduction, convection, and radiation:

$$q_{rtop} := \sigma\cdot A\cdot\left(T_{top}{}^{4} - T_{sky}{}^{4}\right)$$

Radiation. Glass plate to sky.

$$q_{rbottom} := \sigma\cdot A\cdot\left(T_{bottom}{}^{4} - T_{gnd}{}^{4}\right)$$

Radiation. Ground to glass plate.

$$q_{cvtop} := h\cdot A\cdot\left(T_{air} - T_{top}\right)$$

Convection on top side of glass plate.

$$q_{cvbottom} := h\cdot A\cdot\left(T_{air} - T_{bottom}\right)$$

Convection on bottom side of glass plate.

$$q_{cond} := \frac{-k\cdot A\cdot\left(T_{top} - T_{bottom}\right)}{\mathrm{width}}$$

Conduction through glass plate.

Heat exiting the glass plate's top surface must equal the heat entering the glass plate's bottom surface so

$$q_{rtop} + q_{rbottom} := q_{cvtop} + q_{cvbottom} \qquad \text{Eq1}$$

Also, the total heat being conducted through the glass plate must equal the total heat exiting the plate's top surface so

$$q_{cond} := q_{rtop} - q_{cvtop} \qquad \text{Eq2}$$

SOLUTION OF EQ1 AND EQ2 FOR A VERY THIN GLASS PLATE

This solution should give results similar to the results obtained above where we used the equations from reference 3. This will serve as a test of our more general system of equations (Eq1 and Eq2).

So we have two equations (Eq1 and Eq2) and two unknowns (T_{top} and T_{bottom}). We can use MathCad's "solve block" method to solve for T_{top} and T_{bottom}.

The "solve block" method requires an initial "guess" for the value of T_{top} and T_{bottom} so that its iterative convergence routines will converge to a solution. The system of equations comprised of Eq1 and Eq2 has 16 solutions but only one of these solutions contain real numbers that are in the range:

$200 < T_{top} < 300$ and $200 < T_{bottom} < 300$.

(200 Kelvin corresponds to -99.7 F and 300 Kelvin corresponds to 80.3 F)

Since we know the solution should be in that range, a good initial guess for each is 250.

$T_{top} := 250$ Initial guess for T_{top}

$T_{bottom} := 250$ Initial guess for T_{bottom}

Given

$$\sigma \cdot A \cdot \left(T_{top}{}^4 - T_{sky}{}^4\right) + \sigma \cdot A \cdot \left(T_{bottom}{}^4 - T_{gnd}{}^4\right) - h \cdot A \cdot \left(T_{air} - T_{top}\right) - h \cdot A \cdot \left(T_{air} - T_{bottom}\right) = 0$$

$$\frac{-k \cdot A \cdot \left(T_{top} - T_{bottom}\right)}{width} - \sigma \cdot A \cdot \left(T_{top}{}^4 - T_{sky}{}^4\right) + \left[h \cdot A \cdot \left(T_{air} - T_{top}\right)\right] = 0$$

$T_{top} > 200$

$T_{top} < 300$ Restrict solution range

$T_{bottom} > 200$

$T_{bottom} < 300$

$v := \mathrm{Find}\left(T_{top}, T_{bottom}\right)$ Find the solution vector

$v = \begin{pmatrix} 269.869 \\ 269.869 \end{pmatrix}$ $\begin{pmatrix} T_{top} \\ T_{bottom} \end{pmatrix}$

So we now have the Kelvin temperatures of the top and bottom of the extremely thin glass plate in the vector v.

$TtopK := v_0$

$TbottomK := v_1$

We see that for a really thin glass plate, the top surface temperature and the bottom surface Kelvin temperatures (269.87) are equal and match the results of the earlier formula described in reference 3. This should happen if our system of equations is correct so it lends some credence to our general equations.

Converting $T_{top}K$ from Kelvin back to Fahrenheit we find that the top surface of the glass plate will reach the Fahrenheit temperature of:

$$T_{topF1} := KtoF(TtopK)$$

$T_{topF1} = 26.094$ Degrees Fahrenheit!

This explains a lot! With an air temperature of 40 Degrees F, a ground temperature of 45 Degrees F and a nice clear sky with a sky temperature of -70 Degrees C, the surface temperature of a very thin plate will be 6 Degrees F below freezing!

SOLUTION FOR A ONE QUARTER INCH THICK GLASS PLATE

It is a little worse if the glass plate is about the thickness of a telescope corrector plate, which for a 10" telescope tends to be approximately 1/4 inch thick (about .0063 meters)

If we rerun the equations for the case where "width" is .0063 meters then we have the following:

$width := .0063$ (.0063 meters = 1/4 inch)

$T_{top} := 250$ Initial guess for T_{top}

$T_{bottom} := 250$ Initial guess for T_{bottom}

Given

$$\sigma \cdot A \cdot \left(T_{top}^{4} - T_{sky}^{4}\right) + \sigma \cdot A \cdot \left(T_{bottom}^{4} - T_{gnd}^{4}\right) - h \cdot A \cdot \left(T_{air} - T_{top}\right) - h \cdot A \cdot \left(T_{air} - T_{bottom}\right) = 0$$

$$\frac{-k \cdot A \cdot \left(T_{top} - T_{bottom}\right)}{width} - \sigma \cdot A \cdot \left(T_{top}^{4} - T_{sky}^{4}\right) + \left[h \cdot A \cdot \left(T_{air} - T_{top}\right)\right] = 0$$

$T_{top} > 200$

$T_{top} < 300$ Restrict solution range

$T_{bottom} > 200$

$T_{bottom} < 300$

$v := Find\left(T_{top}, T_{bottom}\right)$ Find the solution vector

$$v = \begin{pmatrix} 269.384 \\ 270.352 \end{pmatrix} \qquad \begin{pmatrix} T_{top} \\ T_{bottom} \end{pmatrix}$$

We now have the Kelvin temperatures of the top and bottom of the glass plate in the vector v.

$$TtopK := v_0$$

$$TbottomK := v_1$$

$$TtopK = 269.384$$

$$TbottomK = 270.352$$

We see that the top surface is colder and the bottom surface is warmer than for the case of the very thin glass plate. This is what we would expect since the thicker glass retards the flow of heat.

Converting from Kelvin back to Fahrenheit we find that the top surface of the glass plate will now reach the Fahrenheit temperature of:

$$T_{topF2} := KtoF(TtopK)$$

$T_{topF2} = 25.221$ Degrees Fahrenheit

The temperature of the top surface of the one quarter inch thick glass plate is thus about .9 Degrees F colder than the surface of the very thin glass plate:

$$T_{topF1} - T_{topF2} = 0.872$$

So we have explained with some precision why dew and even ice forms so readily on car windshields and telescope corrector plates when the sky is very clear at night.

Dew will begin to form of course any time the top surface of the glass plate becomes colder than the local dew-point temperature. (The local dew point temperature is typically available from local weather reports. "Weather radio" for example.)

SOLUTION FOR A ONE METER THICK GLASS PLATE!

For a really thick piece of glass, (1 meter thick!), the temperature of the top surface can, for the above temperature parameters, get down to 11.6 degrees F!

An additional 14.5 Degrees F below the quarter inch thick glass plate case!

We rerun the equations for the case where "width" is 1.0 meters:

$width := 1.0$

$T_{top} := 250$ Initial guess for T_{top}

$T_{bottom} := 250$ Initial guess for T_{bottom}

Given

$$\sigma \cdot A \cdot \left(T_{top}^{4} - T_{sky}^{4}\right) + \sigma \cdot A \cdot \left(T_{bottom}^{4} - T_{gnd}^{4}\right) - h \cdot A \cdot \left(T_{air} - T_{top}\right) - h \cdot A \cdot \left(T_{air} - T_{bottom}\right) = 0$$

$$\frac{-k \cdot A \cdot \left(T_{top} - T_{bottom}\right)}{width} - \sigma \cdot A \cdot \left(T_{top}^{4} - T_{sky}^{4}\right) + \left[h \cdot A \cdot \left(T_{air} - T_{top}\right)\right] = 0$$

$T_{top} > 200$

$T_{top} < 300$ Restrict solution range

$T_{bottom} > 200$

$T_{bottom} < 300$

$v := \mathrm{Find}\left(T_{top}, T_{bottom}\right)$ Find the solution vector

$v = \begin{pmatrix} 261.839 \\ 277.683 \end{pmatrix}$ $\begin{pmatrix} T_{top} \\ T_{bottom} \end{pmatrix}$

We now have the Kelvin temperatures of the top and bottom of the glass plate in the vector v.

$TtopK := v_0$

$TbottomK := v_1$

$TtopK = 261.839$

$TbottomK = 277.683$

We see that the top surface is much colder and the bottom surface is much warmer than for the case of the quarter inch glass plate. This is what we would expect since a thick glass plate retards the flow of heat more than a thin one.

Converting from Kelvin back to Fahrenheit we find that the top surface of the glass plate will now reach the Fahrenheit temperature of:

$T_{topF2} := \text{KtoF}(\text{TtopK})$

$T_{topF2} = 11.64$ Degrees Fahrenheit

The temperature of the top surface of the thick glass plate is thus about 14.45 Degrees F colder than the surface of the very thin glass plate:

$T_{topF1} - T_{topF2} = 14.453$

References

1. Heat Transfer, Jack Phillip Holman, McGraw-Hill, 1976, ISBN 0-07-029598-0

2. Heat Transfer, Donald R. Pitts, Leighton E. Sissom, Schaum's Outlines, McGraw-Hill 1997, ISBN 0-07-050207-2

3. www.agr.kuleuven.ac.be/vakken/I106_I125/Ftv4/prob8_12_9.pdf, Heat equation solution for the agricultural problem of a leaf exposed to the night air and sky. (Probably one of the most practical applications.)